REPORT DOCUMENTATION PAGE

Form Approved
OMB NO. 0704-0188

1. AGENCY USE ONLY (Leave blank)

2. REPORT DATE
July, 1997

3. REPORT TYPE AND DATES COVERED
Technical - 97-10

4. TITLE AND SUBTITLE
Asymptotic Theory of the Least Squares Estimators of Sinusoidal Signal

5. FUNDING NUMBERS
DAAH04-96-1-0082

6. AUTHOR(S)
Debasis Kundu

7. PERFORMING ORGANIZATION NAMES(S) AND ADDRESS(ES)
Center for Multivariate Analysis
Dept. of Statistics
417 Thomas Bldg.
Penn State University
University Park, PA 16802

8. PERFORMING ORGANIZATION REPORT NUMBER
97-10

9. SPONSORING / MONITORING AGENCY NAME(S) AND ADDRESS(ES)
U.S. Army Research Office
P.O. Box 12211
Research Triangle Park, NC 27709-2211

10. SPONSORING / MONITORING AGENCY REPORT NUMBER
ARO 35518.14-MA

11. SUPPLEMENTARY NOTES
The views, opinions and/or findings contained in this report are those of the author(s) and should not be construed as an official Department of the Army position, policy or decision, unless so designated by other documentation.

12a. DISTRIBUTION / AVAILABILITY STATEMENT
Approved for public release; distribution unlimited.

12 b. DISTRIBUTION CODE

13 ABSTRACT (Maximum 200 words)

The consistency and the asymptotic normality of the least squares estimators are derived of the sinusoidal model under the assumption of stationary random error. It is observed that the model does not satisfy the standard sufficient conditions of Jennrich (1969), Wu (1981) or Kundu (1991). Recently the consistency and the asymptotic normality are derived for the sinusoidal signal under the assumption of normal error (Kundu; 1993) and under the assumptions of independent and identically distributed random variables in Kundu and Mitra (1996). This paper will generalize them. Hannan (1971) also considered the similar kind of model and establish the result after making the Fourier transform of the data for one parameter model. We establish the result without making the Fourier transform of the data. We give an explicit expression of the asymptotic distribution of the multiparameter case, which is not available in the literature. Our approach is different from Hannan's approach. We do some simulations study to see the small sample properties of the two types of estimators.

14. SUBJECT TERMS
Asymptotic distribution, strong consistency, least squares estimators and stationary distribution.

15. NUMBER IF PAGES
14

16. PRICE CODE

17 SECURITY CLASSIFICATION OR REPORT	18. SECURITY CLASSIFICATION OF THIS PAGE	19. SECURITY CLASSIFICATION OF ABSTRACT	20. LIMITATION OF ABSTRACT
UNCLASSIFIED	UNCLASSIFIED	UNCLASSIFIED	UL

ASYMPTOTIC THEORY OF THE LEAST SQUARES ESTIMATORS OF SINUSOIDAL SIGNAL

Debasis Kundu

Technical Report 97-10

July 1997

Center for Multivariate Analysis
417 Thomas Building
Penn State University
University Park, PA 16802

The research work of the author was partially supported by the Army Research Office under Grant DAAHO4-96-1-0082. The United States Government is authorized to reproduce and distribute reprints for governmental purposes notwithstanding any copyright notation hereon.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

Asymptotic Theory of the Least Squares Estimators of Sinusoidal Signal*

Debasis Kundu
Department of Mathematics
Indian Institute of Technology Kanpur
Pin 208016, India
E-Mail: Kundu@iitk.ernet.in

Abstract

The consistency and the asymptotic normality of the least squares estimators are derived of the sinusoidal model under the assumption of stationary random error. It is observed that the model does not satisfy the standard sufficient conditions of Jennrich (1969), Wu (1981) or Kundu (1991). Recently the consistency and the asymptotic normality are derived for the sinusoidal signal under the assumption of normal error (Kundu; 1993) and under the assumptions of independent and identically distributed random variables in Kundu and Mitra (1996). This paper will generalize them. Hannan (1971) also considered the similar kind of model and establish the result after making the Fourier transform of the data for one parameter model. We establish the result without making the Fourier transform of the data. We give an explicit expression of the asymptotic distribution of the multiparameter case, which is not available in the literature. Our approach is different from Hannan's approach. We do some simulations study to see the small sample properties of the two types of estimators.

Key Words and Phrases: Asymptotic distribution, strong consistency, least squares estimators and stationary distribution.

AMS Subject Classifications (1985): 62J02, 62C05

Short Running Title: Sinusoidal Signal

*This work was partly supported by the Department of Science and Technology, Govt. of India, Grant No. SR/OY/M-06/93.

1 Introduction

The least squares method plays an important role in drawing the inferences about the parameters in the nonlinear regression model. In this paper we consider the least squares estimators (LSE's) of the following sinusoidal time series regression model:

$$Y(t) = A_0 cos(\omega_0 t) + B_0 sin(\omega_0 t) + X(t); \quad t = 1, \ldots, N \tag{1}$$

Here A_0 and B_0 are unknown fixed constants, ω_0 is an unknown frequency lying between 0 and π. X(t)'s are stationary time series satisfying the following assumption:

Assumption 1

$$X(t) = \sum_{j=-\infty}^{\infty} \alpha(j)\epsilon(t-j), \ \sum_{j=-\infty}^{\infty} |\alpha(j)| < \infty \tag{2}$$

where ϵ(t)'s are independent and identically distributed (i.i.d.) random variables with mean zero and finite variance $\sigma^2 > 0$. Here '=' means X(t) has that almost sure representation.

This is an important and well studied model in Time Series and Signal Processing literature. See for example Stoica (1993) for an extensive list of references for different estimation procudures. Hannan (1971, 1973), Walker (1969, 1971), Kundu (1993, 1995), Kundu and Mitra (1995, 1996) also considered this or similar kind of model to study the asymptotic properties of the different estimators and some of the computational issues have been discussed in Rice and Rosenblatt (1988). Walker (1971) considered the approximate least squares estimators (ALSE's) and proved the strong consistency and the asymptotic normaliy of the ALSE's under the assumptions that the errors are i.i.d. random variables with mean zero and finite variance. The result has been extended by Hannan (1971, 1973) to the case when the errors are stationary random variables with continuous spectrum. Kundu (1993) also considered a similar model and proved directly the consistency and the asymptotic normality of the LSE's under the assumption that X(t)'s are i.i.d. with mean zero and finite variance and they are normally distributed. The result was extended to the case of general mean zero and finite variance i.i.d. errors in Kundu and Mitra (1996). In this paper we generalize the result of Kundu and Mitra (1996) to the case when the errors are coming from a mean zero and finite variance stationary process. We prove directly the consistency and the asymptotic normality of the LSE's when the X(t)'s satisfy Assumption 1. It is important to observe that we do not need the continuity assumption of the spectrum. Our approach is straight forward and different from that of Walker (1969, 1971) or Hannan (1971, 1973). Hannan (1971, 1973) obtained the result for the one parameter case after making the Fourier transform of the data. We observe that it is not necessary to make the Fourier transform of the data. We also consider the multiparameter case and obtained the explicit expression of the asymptotic covariance matrix, which is not available in the literature. We also perform some numerical experiments to compare the small sample behavior of the ALSE's and the exact LSE's. In this paper the almost sure convergence means with respect to the usual Lebesgue measure and it will be denoted by a.s.. Also the notation $a = O(N^b)$ means $\left|\frac{a}{N^b}\right|$ is bounded for all N.

The rest of the paper is organised as follows, in Section 2 we prove the consistency of the LSE's and establish the asymptotic normality results in Section 3. The results for the several Harmonic case are obtained in Section 4. Some numerical results are presented in Section 5 and finally we draw conclusions in Section 6.

2 Consistency of the LSE's

Let's denote $\hat{\theta}_N = (\hat{A}_N, \hat{B}_N, \hat{\omega}_N)$ to be the LSE of $\theta_0 = (A_0, B_0, \omega_0)$, obtained by minimizing

$$Q_N(\theta) = \sum_{t=1}^{N} (Y(t) - Acos(\omega t) - Bsin(\omega t))^2 \tag{3}$$

with respect to $\theta = (A, B, \omega)$. It is important to observe that the existence and the uniqueness of a respective measurable function satisfying (3) follows along the same line of Jennrich (1969). To prove the consistency results we need the following lemma.

Lemma 1: Let X(t) be a stationary sequence which satisfies Assumption 1, then

$$\lim_{N\to\infty} \sup_{\theta} \left| \frac{1}{N} \sum_{t=1}^{N} X(t) cos(t\theta) \right| = 0 \text{ a.s.} \tag{4}$$

Before giving the proof in details, we would like to give a sketch of the main idea. First we show that (4) holds for the subsequence N^3. Then we show that

$$\sup_{\theta} \sup_{N^3 < K \leq (N+1)^3} \left| \frac{1}{N^3} \sum_{t=1}^{N} X(t) cos(t\theta) - \frac{1}{K} \sum_{t=1}^{N} X(t) cos(t\theta) \right| \tag{5}$$

converges to zero a.s. as N tends to ∞.

Proof of Lemma 1:

$$\frac{1}{N} \sum_{t=1}^{N} X(t) cos(t\theta) = \frac{1}{N} \sum_{t=1}^{N} \sum_{j=-\infty}^{\infty} \alpha(j) \epsilon(t-j) cos(t\theta) =$$
$$\frac{1}{N} \sum_{t=1}^{N} \sum_{j=-\infty}^{\infty} \alpha(j) \epsilon(t-j) \{cos((t-j)\theta) cos(j\theta) - sin((t-j)\theta) sin(j\theta)\} =$$
$$\frac{1}{N} \sum_{j=-\infty}^{\infty} \alpha(j) cos(j\theta) \sum_{t=1}^{N} \epsilon(t-j) cos((t-j)\theta) -$$
$$\frac{1}{N} \sum_{j=-\infty}^{\infty} \alpha(j) sin(j\theta) \sum_{t=1}^{N} \epsilon(t-j) sin((t-j)\theta) \tag{6}$$

Therefore

$$
\begin{aligned}
&\sup_\theta |\frac{1}{N}\sum_{t=1}^{N} X(t)cos(t\theta)| \leq \\
&\sup_\theta |\frac{1}{N}\sum_{j=-\infty}^{\infty} \alpha(j)cos(j\theta)\sum_{t=1}^{N}\epsilon(t-j)cos((t-j)\theta)| + \\
&\sup_\theta |\frac{1}{N}\sum_{j=-\infty}^{\infty} \alpha(j)sin(j\theta)\sum_{t=1}^{N}\epsilon(t-j)sin((t-j)\theta)| \quad a.s.. \qquad (7)
\end{aligned}
$$

We would like to prove that both the terms on the right hand side of (7) converges to zero as N tends to infinity. Now observe that

$$
\begin{aligned}
&\left\{E\sup_\theta |\frac{1}{N}\sum_{j=-\infty}^{\infty} \alpha(j)cos(j\theta)\sum_{t=1}^{N}\epsilon(t-j)cos((t-j)\theta)|^2\right\}^{\frac{1}{2}} \leq \\
&\frac{1}{N}\sum_{j=-\infty}^{\infty}|\alpha(j)|\left\{E\sup_\theta |\sum_{t=1}^{N}\epsilon(t-j)cos((t-j)\theta)|^2\right\}^{\frac{1}{2}} \leq \\
&\frac{1}{N}\sum_{j=-\infty}^{\infty}|\alpha(j)|\left\{N+\sum_{t=-N+1}^{N} E\left[|\Sigma_m\epsilon(m)\epsilon(m+t)|\right]^{\frac{1}{2}}\right\} \qquad (8)
\end{aligned}
$$

where the sum $\sum_{t=-N+1}^{N}$ omits the term t = 0 and the term Σ_m is over $N-|t|$ term (dependent on j). Since

$$
\sum_{t=-N+1}^{N} E\left[|\Sigma_m\epsilon(m)\epsilon(m+t)|\right] \leq \sum_{t=-N+1}^{N} E\left[|\Sigma_m\epsilon(m)\epsilon(m+t)|^2\right]^{\frac{1}{2}} = O(N^{\frac{3}{2}}) \qquad (9)
$$

(uniformly in j) therefore (8) is $O(N^{-\frac{1}{4}})$. Let $M = N^3$. Therefore

$$
E\sup_\theta |\frac{1}{M}\sum_{j=-\infty}^{\infty}\alpha(j)cos(j\theta)\sum_{t=1}^{N}cos((t-j)\theta)|^2 = O(M^{-\frac{3}{2}}) \qquad (10)
$$

Similarly the result is true if the cosine function is replaced by the sine also. Therefore

$$
\sup_\theta \left|\frac{1}{M}\sum_{t=1}^{M} X(t)cos(t\theta)\right| \to 0 \quad a.s. \qquad (11)
$$

when $M = N^3$. Now

$$
\begin{aligned}
&\sup_\theta \sup_{N^3<K\leq(N+1)^3}\left|\frac{1}{N^3}\sum_{t=1}^{N} X(t)cos(t\theta) - \frac{1}{K}\sum_{t=1}^{N} X(t)cos(t\theta)\right| = \\
&\sup_\theta \sup_{N^3<K\leq(N+1)^3} |\frac{1}{N^3}\sum_{t=1}^{N} X(t)cos(t\theta) - \frac{1}{N^3}\sum_{t=1}^{K} X(t)cos(t\theta) +
\end{aligned}
$$

$$\frac{1}{N^3}\sum_{t=1}^{K} X(t)cos(t\theta) - \frac{1}{K}\sum_{t=1}^{K} X(t)cos(t\theta)| \leq$$

$$\frac{1}{N^3}\sum_{t=N^3+1}^{(N+1)^3} |X(t)| + \sum_{t=1}^{(N+1)^3} |X(t)| \left[\frac{1}{N^3} - \frac{1}{(N+1)^3}\right] \quad a.s.. \tag{12}$$

The mean squared of the first quantity on the right hand side of (12) is dominated by $\frac{K}{N^6}[(N+1)^3 - N^3]^2 = O(N^{-2})$. Similarly the mean squared of the second quantity on the right hand side of (12) is dominated by $K\frac{N^6}{N^8} = O(N^{-2})$. Therefore both will converge to zero almost surely, which proves the lemma.

Corollary 1: The result is true if the cosine function is replaced by the sine function.

Corollary 2: It can be proved similarly that if X(t) is a sequence which satisfies Assumption 1, then

$$\lim_{N\to\infty}\sup_{\theta}\left|\frac{1}{N^3}\sum_{t=1}^{N} t^2 X(t)cos(t\theta)\right| = 0 \text{ a.s.} \tag{13}$$

Now consider

$$\begin{aligned}
&\frac{1}{N}\left[Q_N(\theta) - Q_N(\theta_0)\right] = \\
&\frac{1}{N}\sum_{t=1}^{N}\left\{(Y(t) - Acos(\omega t) - Bsin(\omega t))^2 - X(t)^2\right\} = \\
&\frac{1}{N}\sum_{t=1}^{N}\left(A_0cos(\omega_0 t) + B_0sin(\omega_0 t) - Acos(\omega t) - Bsin(\omega t)\right)^2 + \\
&\frac{2}{N}\sum_{t=1}^{N} X(t)\left(A_0cos(\omega_0 t) + B_0sin(\omega_0 t) - Acos(\omega t) - Bsin(\omega t)\right) = \\
&f_N(A,B,\omega) + g_N(A,B,\omega)
\end{aligned} \tag{14}$$

Now with the help of lemma 1, we can easily conclude that

$$\lim_{N\to\infty}\sup_{\theta\in S_{\delta,M}} g_N(A,B,\omega) = 0 \qquad a.s. \tag{15}$$

where the set $S_{\delta,M}$ for $\delta > 0$, is as follows;

$$\begin{aligned}
S_{\delta,M} &= \{(A,B,\omega), |A - A_0| \geq \delta, |A| \leq M, |B| \leq M \\
&\quad or \quad |B - B_0| \geq \delta, |A| \leq M, |B| \leq M \\
&\quad or \quad |\omega - \omega_0| \geq \delta, |A| \leq M, |B| \leq M\}
\end{aligned} \tag{16}$$

therefore for all $\delta > 0$,

$$\underline{lim}\inf_{S_{\delta},M}\frac{1}{N}\left[Q_N(\theta) - Q_N(\theta_0)\right] = \lim_{N\to\infty}\sup_{\theta\in S_{\delta,M}} f_N(A,B,\omega) > 0. \tag{17}$$

(17) follows easily from Kundu and Mitra (1996). Here $\underline{lim}$ means limit infimum. Now suppose $(\hat{A}_N, \hat{B}_N, \hat{\omega}_N)$ be the LSE's of (A_0, B_0, ω_0) and they are not consistent. Therefore either

Case I: For all subsequences $\{N_K\}$ of {N}, $|\hat{A}_{N_K}| + |\hat{B}_{N_K}|$ tends to infinity or

Case II: There exists a $\delta > 0$ and a $M < \infty$ and a subsequence $\{N_K\}$ such that $(\hat{A}_{N_K}, \hat{B}_{N_K}, \hat{\omega}_{N_K}) \in S_{\delta,M}$, for all K = 1, 2, ...,.

Now

$$Q_{N_K}(\hat{A}_{N_K}, \hat{B}_{N_K}, \hat{\omega}_{N_K}) - Q_{N_K}(A_0, B_0, \omega_0) \leq 0 \tag{18}$$

as $(\hat{A}_{N_K}, \hat{B}_{N_K}, \hat{\omega}_{N_K})$ is the LSE of (A_0, B_0, ω_0), when $N = N_K$. Observe that as $K \to \infty$, for both the cases, the left hand side of (18) converges to a number which is strictly positive, that is a contradiction. Therefore the LSE's of the model (1) have to be strongly consistent. Therefore we can state the following theorem:

Theorem 1 *If $\hat{\theta}_N = (\hat{A}_N, \hat{B}_N, \hat{\omega}_N)$ is the LSE of the nonlinear regression model (1), then it is a strongly consistent estimator of $\theta_0 = (A_0, B_0, \omega_0)$.*

3 Asymptotic Normality

In this section we prove the asymptotic normality of $\hat{\theta}_N$ by using the Taylor series expansion. Let's denote

$$Q'_N(\theta) = \left(\frac{\delta Q_N(\theta)}{\delta A}, \frac{\delta Q_N(\theta)}{\delta B}, \frac{\delta Q_N(\theta)}{\delta \omega}\right) \tag{19}$$

and $Q''_N(\theta)$ to be the corresponding 3×3 matrix which contains the double derivative of $Q_N(\theta)$. Therefore

$$Q'_N(\hat{\theta}_N) - Q'_N(\theta_0) = (\hat{\theta} - \theta_0) Q''_N(\bar{\theta}) \tag{20}$$

where $\bar{\theta} = (\bar{A}, \bar{B}, \bar{\omega})$ is a point in the line joining $\hat{\theta}_N$ and θ_0. Observe that although $\bar{\theta}$ depends on N, we omit it for brevity. Since $Q'_N(\hat{\theta}_N) = 0$, (20) implies

$$(\hat{\theta} - \theta_0) = -Q'_N(\theta_0)\left[Q''_N(\bar{\theta})\right]^{-1} \tag{21}$$

Now

$$\frac{\delta Q_N(\theta_0)}{\delta A} = -2\sum_{t=1}^{N} X(t) cos(\omega_0 t) \tag{22}$$

$$\frac{\delta Q_N(\theta_0)}{\delta B} = -2\sum_{t=1}^{N} X(t) sin(\omega_0 t) \tag{23}$$

$$\frac{\delta Q_N(\theta_0)}{\delta \omega} = -2\sum_{t=1}^{N} tX(t)(A_0 sin(\omega_0 t) - B_0 cos(\omega_0 t)) \tag{24}$$

Also

$$\frac{\delta^2 Q_N(\bar\theta)}{\delta A^2} = 2\sum_{t=1}^{N} cos^2(\bar\omega t), \qquad \frac{\delta^2 Q_N(\bar\theta)}{\delta B^2} = 2\sum_{t=1}^{N} sin^2(\bar\omega t),$$

$$\frac{\delta^2 Q_N(\bar\theta)}{\delta \omega^2} = 2\sum_{t=1}^{N} t^2[(A_0cos(\omega_0 t) + B_0 sin(\omega_0 t) - \bar A cos(\bar\omega t) - \bar B sin(\bar\omega t) + X(t)) \times (\bar A cos(\bar\omega t) + \bar B sin(\bar\omega t)) + (\bar A sin(\bar\omega t) - \bar B cos(\bar\omega t))^2] \tag{25}$$

$$\frac{\delta^2 Q_N(\bar\theta)}{\delta \omega \delta A} = 2\sum_{t=1}^{N} t[sin(\bar\omega t)(A_0cos(\omega_0 t) + B_0 sin(\omega_0 t) - \bar A cos(\bar\omega t) - \bar B sin(\bar\omega t) + X(t)) - cos(\bar\omega t)(\bar A sin(\bar\omega t) - \bar B cos(\bar\omega t))] \tag{26}$$

$$\frac{\delta^2 Q_N(\bar\theta)}{\delta \omega \delta B} = -2\sum_{t=1}^{N} t[cos(\bar\omega t)(A_0cos(\omega_0 t) + B_0 sin(\omega_0 t) - \bar A cos(\bar\omega t) - \bar B sin(\bar\omega t) + X(t)) - sin(\bar\omega t)(\bar A sin(\bar\omega t) - \bar B cos(\bar\omega t))] \tag{27}$$

$$\frac{\delta^2 Q_N(\bar\theta)}{\delta A \delta B} = 2\sum_{t=1}^{N} sin(\bar\omega t)cos(\bar\omega t) \tag{28}$$

Let's define

$$\sigma_{11} = \lim_{N\to\infty} \frac{1}{N}\sum_{t=1}^{N} cos^2(\omega_0 t) = \frac{1}{2}$$

$$\sigma_{22} = \lim_{N\to\infty} \frac{1}{N}\sum_{t=1}^{N} sin^2(\omega_0 t) = \frac{1}{2}$$

$$\sigma_{33} = \lim_{N\to\infty} \frac{1}{N^3}\sum_{t=1}^{N} t^2\left(A_0 sin(\omega_0 t) - B_0 cos(\omega_0 t)\right)^2 = \frac{1}{6}(A_0^2 + B_0^2)$$

$$\sigma_{13} = \sigma_{31} = \lim_{N\to\infty} \frac{1}{N^2}\sum_{t=1}^{N} B_0 t cos^2(\omega_0 t) = \frac{1}{4}B_0$$

$$\sigma_{23} = \sigma_{32} = -\lim_{N\to\infty} \frac{1}{N^2}\sum_{t=1}^{N} A_0 t sin^2(\omega_0 t) = -\frac{1}{4}A_0$$

$$\sigma_{12} = \sigma_{21} = \lim_{N\to\infty} \frac{1}{N}\sum_{t=1}^{N} sin(\omega_0 t)cos(\omega_0 t) = 0$$

Now observe that as $\bar\omega \to \omega_0$, $\bar A \to A_0$ and $\bar B \to B_0$ a.s., we have

$$\lim_{N\to\infty} \frac{1}{N}\sum_{t=1}^{N} cos^2(\bar\omega t) = \lim_{N\to\infty} \frac{1}{N}\sum_{t=1}^{N} cos^2(\omega_0 t) = \frac{1}{2} \tag{29}$$

$$\lim_{N\to\infty} \frac{1}{N}\sum_{t=1}^{N} sin^2(\bar\omega t) = \lim_{N\to\infty} \frac{1}{N}\sum_{t=1}^{N} sin^2(\omega_0 t) = \frac{1}{2} \tag{30}$$

$$\lim_{N\to\infty}\frac{1}{2N^3}\frac{\delta^2 Q_N(\bar\theta)}{\delta\omega^2} = \lim_{N\to\infty}\frac{1}{N^3}\sum_{t=1}^{N} t^2\left(\bar A sin(\bar\omega t) - \bar B cos(\bar\omega t)\right)^2 =$$

$$\lim_{N\to\infty}\frac{1}{N^3}\sum_{t=1}^{N} t^2\left(A_0 sin(\omega_0 t) - B_0 cos(\omega_0 t)\right)^2 = \frac{1}{6}(A_0^2 + B_0^2) \tag{31}$$

$$\lim_{N\to\infty}\frac{1}{2N^2}\frac{\delta^2 Q_N(\bar\theta)}{\delta\omega\delta A} = -\lim_{N\to\infty}\frac{1}{N^2}\sum_{t=1}^{N} t cos(\bar\omega t)\left(\bar A sin(\bar\omega t) - \bar B cos(\bar\omega t)\right) =$$

$$\lim_{N\to\infty}\frac{1}{N^2}\sum_{t=1}^{N} t B_0 cos^2(\omega_0 t) = \frac{1}{4}B_0 \tag{32}$$

$$\lim_{N\to\infty}\frac{1}{2N^2}\frac{\delta^2 Q_N(\bar\theta)}{\delta\omega\delta B} = -\lim_{N\to\infty}\frac{1}{N^2}\sum_{t=1}^{N} t sin(\bar\omega t)\left(\bar A sin(\bar\omega t) - \bar B cos(\bar\omega t)\right) =$$

$$-\lim_{N\to\infty}\frac{1}{N^2}\sum_{t=1}^{N} t A_0 sin^2(\omega_0 t) = -\frac{1}{4}A_0 \tag{33}$$

$$\lim_{N\to\infty}\frac{1}{N}\frac{\delta^2 Q_N(\bar\theta)}{\delta A\delta B} = \lim_{N\to\infty}\frac{2}{N}\sum_{t=1}^{N} sin(\bar\omega t)cos(\bar\omega t) =$$

$$\lim_{N\to\infty}\frac{2}{N}\sum_{t=1}^{N} sin(\omega_0 t)cos(\omega_0 t) = 0 \tag{34}$$

Let's define the 3×3 matrix $\Sigma = ((\sigma_{ij}))$; i,j = 1,2,3 and also define the 3×3 diagonal matrix $\mathbf{D}$ as follows $\mathbf{D}$ = diag $\{ N^{-1/2}, N^{-1/2}, N^{-3/2} \}$. Rewrite (21) as

$$(\hat\theta - \theta_0)\mathbf{D}^{-1} = -Q_N'(\theta_0)\mathbf{D}\left[\mathbf{D}Q_N''(\bar\theta)\mathbf{D}\right]^{-1} \tag{35}$$

Now from (29) - (34) we obtain

$$\lim_{N\to\infty}\mathbf{D}Q_N''(\bar\theta)\mathbf{D} = \lim_{N\to\infty}\mathbf{D}Q_N''(\theta_0)\mathbf{D} = 2\Sigma \tag{36}$$

where

$$\Sigma = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{4}B_0 \\ 0 & \frac{1}{2} & -\frac{1}{4}A_0 \\ \frac{1}{4} & -\frac{1}{4}A_0 & \frac{1}{6}(A_0^2 + B_0^2) \end{bmatrix} \tag{37}$$

and Σ^{-1} exists if $(A_0^2 + B_0^2) > 0$ and it is as follows;

$$\Sigma^{-1} = \frac{4}{A_0^2 + B_0^2}\begin{bmatrix} \frac{1}{2}A_0^2 + 2B_0^2 & -\frac{3}{2}A_0B_0 & -3B_0 \\ -\frac{3}{2}A_0B_0 & \frac{1}{2}B_0^2 + 2A_0^2 & 3A_0 \\ -3B_0 & 3A_0 & 6 \end{bmatrix} \tag{38}$$

Now from the Central Limit theorem of Stochastic Process (see Fuller; 1976) it easily follows that $Q'_N(\theta_0)\mathbf{D}$ tends to a multivariate (3-variate) normal distribution as given below;

$$Q'_N(\theta_0)\mathbf{D} \rightarrow N_3(\mathbf{0}, 4\sigma^2 c\Sigma) \tag{39}$$

where

$$c = |\sum_{j=-\infty}^{\infty} \alpha(j)cos(\omega_0 j)|^2 + |\sum_{j=-\infty}^{\infty} \alpha(j)sin(\omega_0 j)|^2$$

Therefore we have;

$$(\hat{\theta}_N - \hat{\theta}_0)\mathbf{D}^{-1} \rightarrow N_3(\mathbf{0}, \sigma^2 c\Sigma^{-1}) \tag{40}$$

Now we can state the result as the following theorem;

Theorem 2 *Under the assumptions of Theorem 1,* $\{N^{\frac{1}{2}}(\hat{A}_N - A_0)\ , N^{\frac{1}{2}}(\hat{B}_N - B_0), N^{\frac{3}{2}}(\hat{\omega}_N - \omega_0)\}$ *converges in distribution to a 3-variate normal distribution with mean vector zero and the dispersion matrix is given by* $\sigma^2\ c\ \Sigma^{-1}$*, where* c *and* Σ^{-1} *are as defined before.*

4 Multiparameter Case

In this section we will extend the results of Section 2 and Section 3 to the following model:

$$Y(t) = \sum_{K=1}^{M} A_0^K cos(\omega_0^K t) + B_0^K sin(\omega_0^K t) + X(t); \quad t = 1, \ldots, N, \tag{41}$$

where A_0^K, B_0^K are arbitrary real numbers and ω_0^K's are the distinct frquencies lying between 0 and π for K = 1, ..., M. X(t)'s satisfy Assumption 1.

Let us use the following notations $\mathbf{A} = (A^1, \ldots, A^M)$, $\mathbf{B} = (B^1, \ldots, B^M)$ and $\omega = (\omega^1, \ldots, \omega^M)$. Similarly $\mathbf{A_0}$, $\mathbf{B_0}$, $\omega_\mathbf{0}$ and $\mathbf{\hat{A}_N}$, $\mathbf{\hat{B}_N}$ and $\hat{\omega}_\mathbf{N}$ are also defined. We would like to investigate the consistency and the asymptotic normality properties of the LSE's obtained by minimizing $R_N(\Phi) =$,

$$\sum_{t=1}^{N} \left(Y(t) - \sum_{K=1}^{M} [A^K cos(\omega^K t) + B^K sin(\omega^K t)] \right)^2$$

with respect to $\Phi = (\mathbf{A}, \mathbf{B}, \omega)$. Now we have the following result:

Theorem 3 *If* $\hat{\Phi}_N = (\hat{\mathbf{A}}_N, \hat{\mathbf{B}}_N, \hat{\omega}_N)$ *is the LSE of* $\Phi_0 = (\mathbf{A}_0, \mathbf{B}_0, \omega_0)$*, then* $\hat{\Phi}_N$ *is a strongly consistent estimator of* $\hat{\Phi}_0$*.*

Proof of Theorem 3: With the help of Lemma 1 and using the similar kind of techniques as that of (Kundu and Mitra; 1995), the results can be established.

Let's denote the 1×3M vector $R'_N(\Phi)$ as follows:

$$R'_N(\Phi) = \left(\frac{\delta R_N(\Phi)}{\delta \mathbf{A}}, \frac{\delta R_N(\Phi)}{\delta \mathbf{B}}, \frac{\delta R_N(\Phi)}{\delta \omega} \right)$$

and $R''_N(\Phi)$ denotes the $3M \times 3M$ matrix which contains the double derivative of $R_N(\Phi)$. Now we have

$$R'_N(\hat{\Phi}_N) - R'_N(\Phi_0) = (\hat{\Phi}_N - \Phi_0) R''_N(\bar{\Phi}) \tag{42}$$

where $\bar{\Phi} = (\bar{\mathbf{A}}, \bar{\mathbf{B}}, \bar{\omega})$ is a point in the line joining $\hat{\Phi}_N$ and Φ_0. Since $R'_N(\hat{\Phi}_N) = 0$, we have

$$(\hat{\Phi}_N - \Phi_0) = -R'_N(\Phi_0) \left[R''_N(\bar{\Phi}) \right]^{-1} \tag{43}$$

Let's define the $3M \times 3M$ diagonal matrix $\mathbf{V}$ whose first 2M diagonal elements are $N^{-\frac{1}{2}}$ and the last M diagonal elements are $N^{-\frac{3}{2}}$. Therefore we can write (43) as

$$(\hat{\Phi}_N - \Phi_0)\mathbf{V}^{-1} = -R'_N(\Phi_0)\mathbf{V}^{-1} \left[\mathbf{V}^{-1} R''_N(\bar{\Phi}) \mathbf{V}^{-1} \right]^{-1}$$

Now using the similar kind of arguments as of Section 3, we can say that

$$R'_N(\Phi_0)\mathbf{V} \rightarrow N_{3M}(\mathbf{0}, 4\sigma^2 \mathbf{G})$$

where $\mathbf{G}$ is a $3M \times 3M$ matrix and it has the following structure

$$\mathbf{G} = \begin{bmatrix} \mathbf{G_{11}} & \mathbf{G_{12}} & \mathbf{G_{13}} \\ \mathbf{G_{21}} & \mathbf{G_{22}} & \mathbf{G_{23}} \\ \mathbf{G_{31}} & \mathbf{G_{32}} & \mathbf{G_{33}} \end{bmatrix} \tag{44}$$

where each of the $\mathbf{G_{ij}}$ is a $M \times M$ matrix and

$$\begin{aligned} \mathbf{G_{11}} &= \mathbf{G_{22}} = diag\left\{ \frac{1}{2}c_1, \ldots, \frac{1}{2}c_M \right\} \\ \mathbf{G_{13}} &= \mathbf{G_{31}} = diag\left\{ \frac{1}{4}B_0^1 c_1, \ldots, \frac{1}{2}B_0^M c_M \right\} \\ \mathbf{G_{23}} &= \mathbf{G_{32}} = -diag\left\{ \frac{1}{4}A_0^1 c_1, \ldots, \frac{1}{2}A_0^M c_M \right\} \\ \mathbf{G_{33}} &= \frac{1}{6} diag\left\{ d_1, \ldots, d_M \right\} \\ \mathbf{G_{12}} &= \mathbf{0}. \end{aligned} \tag{45}$$

here $c_K =$

$$|\sum_{j=-\infty}^{\infty} \alpha(j) cos(\omega_0^K j)|^2 + |\sum_{j=-\infty}^{\infty} \alpha(j) sin(\omega_0^K j)|^2$$

and $d_K = c_K \left[(A_0^K)^2 + (B_0^K)^2\right]$ for K = 1, ..., M. Observe that

$$\lim_{N\to\infty} \mathbf{V} R_N''(\bar{\Phi})\mathbf{V} = \lim_{N\to\infty} \mathbf{V} R_N''(\Phi_0)\mathbf{V} = 2\Gamma \tag{46}$$

here the $3M \times 3M$ matrix Γ is

$$\Gamma = \begin{bmatrix} \frac{1}{2}\mathbf{I_M} & \mathbf{0} & \mathbf{S_1} \\ \mathbf{0} & \frac{1}{2}\mathbf{I_M} & \mathbf{S_2} \\ \mathbf{S_1} & \mathbf{S_2} & \mathbf{S_3} \end{bmatrix} \tag{47}$$

where $\mathbf{S_1}, \mathbf{S_2}, \mathbf{S_3}$ are $M \times M$ diagonal matrices as follows;

$$\begin{aligned} \mathbf{S_1} &= \frac{1}{4} diag\left\{B_0^1, \ldots, B_0^M\right\} \\ \mathbf{S_2} &= -\frac{1}{4} diag\left\{A_0^1, \ldots, A_0^M\right\} \\ \mathbf{S_3} &= \frac{1}{6} diag\left\{d_1, \ldots, d_M\right\} \end{aligned} \tag{48}$$

and $\mathbf{I_M}$ is the identity matrix of order M. Since

$$\Gamma^{-1} = 4 \begin{bmatrix} \frac{1}{2}\mathbf{R_4} + 2\mathbf{R_1} & \mathbf{R_2} & \mathbf{R_3} \\ \mathbf{R_2} & \frac{1}{2}\mathbf{R_1} + 2\mathbf{R_4} & \mathbf{R_5} \\ \mathbf{R_3} & \mathbf{R_5} & \mathbf{R_6} \end{bmatrix} \tag{49}$$

where

$$\begin{aligned} \mathbf{R_1} &= diag\left\{\frac{(B_0^1)^2}{d_1}, \ldots, \frac{(B_0^M)^2}{d_M}\right\} \\ \mathbf{R_2} &= -\frac{3}{2} diag\left\{\frac{A_0^1 B_0^1}{d_1}, \ldots, \frac{A_0^M B_0^M}{d_M}\right\} \\ \mathbf{R_3} &= -3 diag\left\{\frac{B_0^1}{d_1}, \ldots, \frac{B_0^M}{d_M}\right\} \\ \mathbf{R_4} &= diag\left\{\frac{(A_0^1)^2}{d_1}, \ldots, \frac{(A_0^M)^2}{d_M}\right\} \\ \mathbf{R_5} &= 3 diag\left\{\frac{A_0^1}{d_1}, \ldots, \frac{A_0^M}{d_M}\right\} \\ \mathbf{R_6} &= 6 diag\left\{\frac{1}{d_1}, \ldots, \frac{1}{d_M}\right\} \end{aligned} \tag{50}$$

we have

$$(\hat{\Phi}_N - \Phi_0)\mathbf{V}^{-1} \to N_{3M}\left(\mathbf{0}, \sigma^2 \Gamma^{-1} \mathbf{G} \Gamma^{-1}\right)$$

therefore we can state the result as the following theorem;

Theorem 4 *Under the assumptions of Theorem 3,* $\{N^{\frac{1}{2}}(\hat{\mathbf{A}}_N - \mathbf{A}_0)$ $,N^{\frac{1}{2}}(\hat{\mathbf{B}}_N - \mathbf{B}_0)$, $N^{\frac{3}{2}}(\hat{\omega}_N - \omega_0)\}$ *converges in distribution to a 3M-variate normal distribution with mean vector zero and the dispersion matrix is given by* $\sigma^2\Gamma^{-1}\mathbf{G}\Gamma^{-1}$.

5 Numerical Experiments

In this section we perform some Monte Carlo simulations to see how the asymptotic results work for small sample. We considered the following model:

$$Y(t) = A_0 cos(\omega_0 t) + B_0 sin(\omega_0 t) + X(t); \quad t = 1, \ldots, N. \tag{51}$$

We took $A_0 = B_0 = 1.5$, $\omega_0 = .25\,\pi$ (≈ 0.735398), $.50\,\pi$ (≈ 1.570796) and $.75\,\pi$ (≈ 2.356194). X(t) = ϵ(t) + .5ϵ(t-1), where ϵ(t)'s are i.i.d. normal random variables with mean zero and variance one. Numerical results are reported for N = 10, 15, 25. All these computations were performed at the Indian Instituet of Technology Kanpur, using PC-486 and the random deviate generator proposed by Press et al. (1992). For a particular N and ω, 1000 different data sets were generated and for each data set we estimated the nonlinear parameters by two different methods, one (denoted by L.S.) by directly minimizing (3) with respect to the different parameters and the other one (denoted by A.L.S.) by first making the Fourier transform of the data as suggested by Hannan (1971, 1973), Walker (1971). We computed the average estimates and the average mean squared errors over 1000 replications. We reported the result in Table 1 for the frequency only because the others are quite similar in nature. The figures in the top denote the average estimates and the figures in the parenthesis below give the corresponding averge mean squared errors. We also computed the 95% confidence interval for ω for each data sets. The results are reported in Table 2. The first figure in the parenthesis is the average length of the confidence interval and the second figure is the coverage frequency over 1000 replications. From Table 1 and Table 2, it is clear that although asymptotically both the methods are same but for small sample it is observed that the exact LSE's are better than the ALSE's. The average mean squared errors of ω are lower for the usual LSE's for almost all the sample sizes and for all ω's. About the confidence intervals, it is observed that for higher values of ω, the confidence interval of ω obtained by using the exact LSE's usually give higher coverage probability. It is also observed that for both the methods as N increases the average length decreases and the coverage probability increases.

6 Conclusions

In this paper we considered the one parameter and multiparameter sinusoidal model under the assumption of additive stationary errors. We obtained the asymptotic properties of the

LSE's directly without making the Fourier transform of the data. We also obtained the explicit expression of the covariance matrix for the multiparameter case, which is not available in the literature. From the numerical study it is observed that although asymptotically the two mehtods are equivalent but the exact LSE's are better than the ALSE's in terms of the mean squared errors. Since both the methods require the same amount of computaions, therefore it is recommended not to Fourier transform the data at least for small samples to make any finite sample inference from the asymptotic results.

Acknowledgements:

The author would like to thank the referee for many valuable suggestions.

REFERENCES:

[1] Fuller, W.A. (1976),*Introduction to Statistical Time Series*, John Wiley and Sons, New York.

[2] Hannan, E.J. (1971),"Nonlinear Time Series Regression", *Journal of Applied Probability*, vol. 8, pp 767-780.

[3] Hannan, E.J. (1973), "The Estimation of Frequencies", *Journal of Applied Probability*, vol. 10, pp 510-519.

[4] Jennrich, R.I. (1969), "Asymptotic Properties of Non Linear Least Squares Estimation", *Annals of Mathematical Statistics*, vol. 40, pp 633-643.

[5] Kundu, D. (1991), "Asymptotic Properties of the Complex Valued Nonlinear Regression Model", *Communications in Statistics, Theory and Methods*, vol. 24, pp 3793-3803.

[6] Kundu, D. (1993), "Asymptotic Theory of the Least Squares Estimator of a Particular Nonlinear Regression Model", *Statistics and Probablity Letters*, vol. 18, pp 13-17.

[7] Kundu, D. (1995) "Consistency of the Undamped Exponential Signals Model on a Restricted Parameter Space", *Communication in Statsitics, Theory and Methods*, vol. 24, pp 241-251.

[8] Kundu, D. and Mitra, A. (1995), "A Note on the Consistency of the Undmped Exponetial Signals Model", *Statistics*, vol. 26, pp 1-9.

[9] Kundu, D. and Mitra, A. (1996), "Asymptotic Theory of the Least Squares Estimators of a Nonlinear Time Series Regression Model", *Communications in Statsitics, Theory and Methods*, vol. 25, no. 1, pp 133-141.

[10] Press, W.H., Teukolsky, S.A., Vellerling, W.T. and Flannery, B.P. (1992), *Numerical Recipies in Fortran, The Art of Scientific Computing*, 2nd. Edition, Cambridge University Press.

[11] Rice, J.A. and Rosenblatt, M. (1988), "On Frequency Estimation", *Biometrika*, vol. 75, pp 477-484.

[12] Stoica, P. (1993), "List of References on Spectral Estimation", *Signal Processing*, vol. 31, pp 329-340.

[13] Walker, A.M. (1969), "On the Estimation of a Harmonic Components in a Time Series with Stationary Residuals", *Proceedings of the International Statistical Institute*, vol. 43, pp 374-376.

[14] Walker, A.M. (1971), "On the Estimation of a Harmonic Components in a Time Series with Stationary Independent Residuals", *Biometrika*, vol. 58, pp21-26.

[15] Wu, C.F.J. (1981), "Asymptotic Theory of Non Linear Least Squares Estimation", *Annals of Statistics*, vol. 9, pp 501-513.

Table 1

ω	N = 10		N = 15		N = 25	
	L.S.	A.L.S.	L.S	A.L.S.	L.S.	A.L.S.
$.25\pi$	.7093	.6949	.7525	.7190	.7871	.7794
	(.1314)	(.1367)	(.0287)	(.0539)	(.0078)	(.0139)
$.50\pi$	1.3402	1.2555	1.4371	1.4543	1.4749	1.4497
	(.3387)	(.5072)	(.1287)	(.1467)	(.0975)	(.1436)
$.75\pi$	1.7772	1.6292	2.1143	2.0450	2.2501	2.1875
	(1.060)	(1.426)	(.4455)	(.6202)	(.1987)	(.3637)

Table 2

ω	N = 10		N = 15		N = 25	
	L.S.	A.L.S.	L.S	A.L.S.	L.S.	A.L.S.
$.25\pi$	(.24, 46)	(.24, .41)	(.25, .73),	(.15, .53)	(.15, .88),	(.09, .81)
$.50\pi$	(.31, 53)	(.21, .42)	(.29, .78),	(.19, .71)	(.16, .87),	(.10, .73)
$.75\pi$	(.35, 62)	(.37, .49)	(.32, .86),	(.21, .80)	(.17, .94),	(.11, .86)